Exam Quantum Field Theory
January 25, 2016
Start: 9:00h End: 12:00h

Each sheet with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you on one side, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also loose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

The energy projectors for spin 1/2 Dirac fermions:

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \frac{\not{p}+m}{2m}$$

$$\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) = \frac{\not{p}-m}{2m}$$

$$\{\gamma_5, \gamma_\mu\} = 0 \qquad \gamma_5^2 = \mathbb{1}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad (d=4)$$

1. (3 points total) Given the lagrangian density for two interacting real scalar fields $\phi_{1,2}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2 - \frac{\lambda}{2}\phi_1\phi_2^2$$

a) [**1.5 points**] Derive the Feynman rule for the interaction vertex and draw the corresponding Feynman diagram in momentum space (solid line for ϕ_1, dash line for ϕ_2). **Show your work**

b) [**1.5 points**] One of the possible processes in this theory is the scattering of particle 1 and particle 2, $1\,2 \to 1\,2$. By using the Feynman rules for this theory in momentum space, draw the Feynman diagrams contributing to the leading order (in λ) non-zero contributions to the scattering amplitude $1\,2 \to 1\,2$, and write the corresponding amplitudes in terms of the Mandelstam variables s, t, u. ***Note:*** *You do not need to explicitly derive the Feynman rules for the free scalar theory.*

2. (3 points total) Consider the QED lagrangian density with the addition of the Pauli term in d spacetime dimensions,

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c\bar{\psi}\sigma^{\mu\nu}\psi\, F_{\mu\nu}$$

with $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$, m the fermion mass, $F_{\mu\nu}$ the fully antisymmetric electromagnetic field strenght tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, c the coupling of the Pauli term and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$.

a) [**1 point**] Find how each term in the lagrangian transforms under the chiral transformation

$$\begin{aligned} \psi'(x) &= e^{i\alpha\gamma_5}\,\psi(x) \\ \bar{\psi}'(x) &= \bar{\psi}(x)e^{i\alpha\gamma_5} \\ A'_\mu(x) &= A_\mu(x) \end{aligned}$$

with α a real parameter of the transformation. **Show your work**

b) [**1 point**] Compute the canonical dimension of the coupling c of the Pauli term. **Show your work**

c) [**1 point**] For which spacetime dimension d is the theory with the Pauli term renormalizable? super-renormalizable? non-renormalizable?

Hint: In your argument you may also want to make use of the general formula for the superficial degree of divergence for QED with the inclusion of QED-like vertices $\sim \bar{\psi}\psi A$ *that contain additional derivatives (e.g. Pauli term): In this case*

$$\delta(D) = d + \sum_n \left(\frac{d-4}{2} + n\right) V_n - \left(\frac{d-1}{2}\right) E_F - \left(\frac{d-2}{2}\right) E_B$$

where V_0 *(*$n = 0$*) is the number of QED vertices,* V_1 *(*$n = 1$*) the number of QED-like vertices with one additional derivative,* V_2 *(*$n = 2$*) with two additional derivatives etc.*

4. (3 points total) Electrons can interact with an external classical electromagnetic field as shown in Figure 1, in which the source of the field is represented by a cross. Assume that the electromagnetic field is static, i.e. $q_0 = 0$. In other words, its recoil energy is negligible and the scattering of the electron is elastic $|\vec{p}| = |\vec{p}'|$.

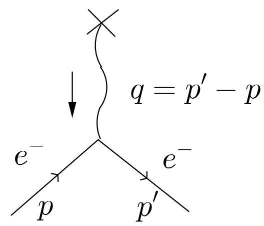

Figure 1: Scattering of an electron by an external electromagnetic field.

a) **[1 point]** By using the Feynman rules of QED in momentum space, write the scattering amplitude for the diagram in Figure 1 in the case of a Coulomb field of a heavy nucleus with atomic number Z

$$A^{\mu}_{e.m.}(\vec{q}) = \left(\frac{Ze}{|\vec{q}|^2},\ 0,\ 0,\ 0 \right)$$

Hint: The Feynman rule for the external classical electromagnetic field line amounts to the factor $A^{\mu}_{e.m.}(\vec{q})$*.*

b) **[2 points]** Derive the unpolarized differential cross-section

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi} \right)^2 X$$

for the scattering process in Figure 1, with X the unpolarized squared amplitude. In particular, and by using relativistic kinematics, show that it is given by

$$\frac{d\sigma}{d\Omega} = \frac{(\alpha Z)^2}{4E^2 v^4 \sin^4(\theta/2)} \left(1 - v^2 \sin^2(\theta/2)\right)$$

with $\alpha = e^2/(4\pi)$ the fine structure constant, E the energy of the electron, v its velocity and θ the scattering angle (between the outgoing and incoming electron). **Show your work**